INTRODUCTION

CHAPTER I

Algebraic Analysis before Al-Khowarizmi

Arabic contributions to science have, in the past, been somewhat neglected by historians. More recent studies are recognizing our indebtedness to Mohammedan scholars, who kept the embers of learning aglow while Europe was in the darkness of the Middle Ages. Much of our knowledge of Greek mathematics comes to us from Arabic sources; the early Latin versions were frequently based upon Arabic texts rather than the Greek originals. Similarly, Hindu arithmetic and astronomy were transmitted to Europe by Islam. The services of the Arabs to science were not limited to the preservation and transmission of the learning of other nations. They made independent contributions in many fields.

Among these achievements is the Arabic algebra of Al-Khowarizmi, which for centuries enjoyed wide popularity in the original, and for further centuries extended its popularity through translations and adaptations. A study of the content of this work is an excursion into mediæval thought. By a study of the text in a form as nearly like the original as possible, we discover the reason for its long-continued appeal to the Occidental as well as the Oriental mind, its interest for the Englishman, the German, and the Italian, as well as for the Arab. Even to-day teachers of elementary mathematics may find this book fruitful in suggestion: the geometric solutions of quadratic equations presented by the Arabic writer more than a thousand years ago may be used with profit in our classrooms.

Simple equations of the first degree in one unknown, of the type $ax=b$, are found in the oldest mathematical text-book which we possess, the Ahmes papyrus of about 1700 B.C., which was published with a German translation by Eisenlohr.\footnote{A. Eisenlohr, Ein mathematisches Handbuch der alten Ägypter, Leipzig, 1877; Fascimile of the Rhind Mathematical Papyrus in the British Museum, with preface by E. A. Wallis Budge, London, 1898.} This Egyp-
tian document presents not only first degree equations together with symbols for the unknown quantity and for the operations of addition and subtraction, but also shows traces of a study of simultaneous linear equations some two thousand years before the Christian Era. Later, but still before the golden age of Greek mathematics, the quadratic equation appears in Egypt. The problems found involve simultaneous quadratic equations, thus: 1

"Another example of the distribution of a given area into squares. If you are told to distribute 100 square ells (units of area) over two squares so that the side of one shall be \( \frac{3}{4} \) of the side of the other: please give me each of the unknowns." The solution follows by assuming the side of one square to be unity, and the other \( \frac{3}{4} \). The sum of these areas is \( \frac{7}{4} \), of which the root is \( \frac{5}{4} \). The root of 100 is 10; 10, then, is to the required side as \( \frac{5}{4} \) is to 1, whence one side is 8 and the other 6. The algebraical equivalent of this geometrical problem is, evidently,

\[
x^2 + y^2 = 100,
\]

\[
y = \frac{3}{4} x.
\]

Noteworthy also is the fact that a symbol for square root occurs in the discussion of these problems.

The solution above leads to the number relation, \( 6^2 + 8^2 = 10^2 \), which connects directly with the simpler form, \( 3^2 + 4^2 = 5^2 \), and to the same relation other problems of this kind reduce. 2 This makes connection, of course, with the so-called Pythagorean theorem that the sum of the squares on the sides of a right triangle equals the square on the hypotenuse. Even though the Egyptians had no logical proof for this proposition, their familiarity with it is well established. In the time of Plato, and for some centuries afterwards, the Egyptians were famed as surveyors, and the principle stated seems to have been applied by them in laying out right angles by means of a long rope knotted at equal intervals. Two pegs situated three units apart are set out along the line to which it is desired to draw a perpendicular. From one peg

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1 M. Cantor, Vorlesungen über Geschichte der Mathematik, Vol. I, third edition (Leipzig, 1907), pp. 95–96. To this work we shall refer as Cantor, I (3), and to the other volumes similarly.


an arc is swung with a radius of four units, while from the other end an arc is swung with a radius of five units. The intersection is connected with the peg from which the shorter arc is swung, forming thus a right angle with the desired line, for in any triangle with sides in the ratio three to four to five, a right angle lies opposite the longest side.

The Pythagorean theorem was applied also in India, before the time of Pythagoras, in the construction of altars. With this theorem as developed in the Apastamba Sulba Sutras, the rules for altar construction, are associated careful approximations of square root, pure quadratic equations, and even, as Milhau has shown, the possible solution of the complete quadratic equation,

\[ ax^2 + bx = c. \]

The ancient Babylonians, furthermore, constructed tables of squares and cubes. Such tables are found upon the famous tablets of Senkereh, which are contemporary with the Ahmes papyrus. Application of these quadratic numbers to problems similar to those of Egypt already mentioned has not been discovered, but the fact is evident that such tables were a step toward the study of quadratic equations. Cantor shows that the ancient Hebrews were probably familiar with the 3, 4, 5 right triangle. In China, too, students mathematically inclined had come upon this number relation, and evidently were studying quadratic numbers.

Familiarity of Greek mathematicians with the geometrical solution of quadratic equations in the time of Pythagoras is now well established. Hippocrates (fifth century B.C.) writing on the quadrature of the lunes, in an attempt to square the circle, assumes a construction which is equivalent to the solution of the equation,

\[ x^2 + \sqrt{3/2}ax = a^2. \]


4 Cantor, I (3), pp. 49.


Several propositions of Euclid present geometrical equivalents of the solution of various types of quadratic equations, not involving negative coefficients, and further study of similar problems appears in Euclid’s *Data*. Of this nature are the fifth, sixth, and eleventh propositions of the second book of the *Elements* and the twenty-seventh, twenty-eighth, and twenty-ninth of the sixth book, and problems 84, 85, 86, and others of the *Data*.\(^1\) Problem 84, for example, reads:

“If two straight lines include a given area in a given angle, and the excess of the greater over the less is given, then each of them is given.”

This corresponds to the equations:

\[
xy = k^2 \\
x - y = a.
\]

The two following problems (85 and 86) correspond to the simultaneous quadratic equations:

\[
xy = k^2, \\
x + y = a, \\
xy = k^2, \\
x^2 - y^2 = a^2.
\]

The eleventh proposition of the second book of the *Elements* furnishes the solution of the equation

\[
x^2 + ax = a^2
\]
or even more general,

\[
x^2 + ax = b^2.
\]

As this so well illustrates the geometrical solution, it is given in full, following Heath’s *Euclid*.

**Book II of the Elements of Euclid, Proposition 11**

\(^{1}\)References and citations from the *Elements* are to Heath’s *Euclid* and the *Data* (Greek and Latin) edited by H. Menge, Leipzig, 1896, being Vol. VI of *Euclidis opera omnia*, ed. Heiberg et Menge. An English translation of the *Data* is found in the numerous editions of *The Elements of Euclid* by Simson. The numbering of the problems is slightly different in the two versions.
"For let the square $ABDC$ be described on $AB$ (I. 46); let $AC$ be bisected at the point $E$, and let $BE$ be joined; let $CA$ be drawn through to $F$, and let $EF$ be made equal to $BE$; let the square $FH$ be described on $AF$, and let $GH$ be drawn through to $K$.

"I say that $AB$ has been cut at $H$ so as to make the rectangle contained by $AB$, $BH$ equal to the square on $AH$.

"For, since the straight line $AC$ has been bisected at $E$, and $FA$ is added to it, the rectangle contained by $CF$, $FA$ together with the square on $AE$ is equal to the square on $EF$. (II. 6.)

"But $EF$ is equal to $EB$; therefore the rectangle $CF$, $FA$ together with the square on $AE$ is equal to the square on $EB$.

"But the squares on $BA$, $AE$ are equal to the square on $EB$, for the angle at $A$ is right (I. 47); therefore the rectangle $CF$, $FA$ together with the square on $AE$ is equal to the squares on $BA$, $AE$.

"Let the square on $AE$ be subtracted from each; therefore the rectangle $CF$, $FA$ which remains is equal to the square on $AB$.

"Now the rectangle $CF$, $FA$ is $FK$, for $AF$ is equal to $FG$; and the square on $AB$ is $AD$; therefore $FK$ is equal to $AD$.

"Let $AK$ be subtracted from each; therefore $FH$ which remains is equal to $HD$.

"And $HD$ is the rectangle $AB$, $BH$, for $AB$ is equal to $BD$; and $FH$ is the square on $AH$; therefore the rectangle contained by $AB$, $BH$ is equal to the square on $HA$.

"Therefore the given straight line $AB$ has been cut at $H$ so as to make the rectangle contained by $AB$, $BH$ equal to the square on $HA$. Q. E. F."

The ordinary algebraical solution of the corresponding equation

$$x^2 + ax = a^2,$$

from $a(a-x) = x^2$,

parallels this geometrical demonstration.

To complete the square in the left-hand member, $\frac{a^2}{4}$ is added to both members. This corresponds to marking the point $E$ on the figure, for then the square on $BE$ equals $a^2 + \frac{a^2}{4}$ or $AB^2 + AE^2$.

Extracting the square root of both members, we have, algebraically,

$$x + \frac{a}{2} = \pm \sqrt{\frac{5a^2}{4}},$$

the negative sign being disregarded. The right-hand member corresponds to the line $BE$ and the left-hand member to $EF$, which is equal to $BE$. 


Algebraically we proceed by subtracting $\frac{a}{2}$ from both members, giving
\[ x = \sqrt{\frac{5a^2}{4} - \frac{a}{2}}. \]

This corresponds to the line $AF$ in the figure which is $BE - AE$.

Analytical solution of the quadratic equation appears quite definitely in the works of Heron of Alexandria, who flourished about the beginning of the Christian Era. Heron states in effect that given the sum of two line segments and their product then each of the segments is known. However, he goes farther than any work of Euclid in applying this to a numerical example,
\[ 144x(14 - x) = 6720. \]

Without putting this into the form of an equation, Heron states that the approximate value of $x$ is $8\frac{1}{2}$, and this evidently indicates an analytical solution. The geometrical garb is absolutely discarded in a problem in the *Geometry* doubtfully attributed to Heron. The problem is to compute the diameter of a circle given the sum of the area, the circumference, and the diameter, summing an area and lengths, entirely contrary to Greek usage. The form of the result, practically
\[ x = \sqrt{\frac{154 \times 212 + 841}{11}} - 29, \]
indicates that the equation
\[ \frac{11}{14}x^2 + \frac{23}{7}x = 212 \]
was put in the form
\[ 121x^2 + 638x = (212)(154). \]

Somewhat similar problems in which lines and areas are summed appear in Greece in the period between Heron and Diophantus (about 250 A.D.) as well as in the works of the latter. One of these problems is to find a square whose area and perimeter together equal 896 ($x^2 + 4x = 896$). The solution pro-

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ceeds in the ordinary manner by adding to 896 the square of half the coefficient of \( x \) and then taking the square root of this sum. From this is subtracted one-half the coefficient of \( x \), giving the side 28. Four other problems of this series deal with right triangles having rational sides and hypotenuse, in which the sum of the area and perimeter is to equal a given number. If \( a, b \) are the sides, \( c \) the hypotenuse, \( S \) the area, \( r \) radius of the inscribed circle, and \( s = \frac{1}{2}(a + b + c) \), then the solution depends upon the following formulae:

\[
S = rs = \frac{1}{2}ab, \quad r + s = a + b, \quad c = s - r.
\]

\[
a = \frac{r + s \pm \sqrt{(r + s)^2 - 8rs}}{2}.
\]

In the sixth book of the *Arithmetica* Diophantus treats rational right triangles in which the area plus or minus one side is a given number, or the area plus or minus the sum of two sides or one side and the hypotenuse, is a given number. Again, such a problem appears in an algebraic work by Shojā ben Aslam, Abū Kāmil, an Arabic writer of the tenth century.\(^1\)

In respect to analysis Diophantus is the greatest name among the Greeks. Recently it has been established that he flourished in the third century of the Christian Era, when Greek supremacy in mathematics was waning. No doubt whatever exists that this Alexandrian was familiar with the analytical solution of the various forms of quadratic equations, neglecting negative roots and, of course, imaginary roots, which did not receive serious treatment for more than a millennium after Diophantus. The three types of complete quadratic equations, involving only positive coefficients, are the following:

\[
ax^2 + bx = c,
\]

\[
ax^2 + c = bx,
\]

\[
ax^2 = bx + c.
\]

All three types appear in the *Arithmetica* of Diophantus, not systematically treated but solved as incidental to the solution of other problems. In fact, after dealing with the solution of equations of the form

\[
ax^n = bx^n,
\]

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Diophantus makes the following explicit statement regarding his intention of writing a systematic treatise on the quadratic equation: ¹

"This should be the object aimed at in framing the hypotheses of propositions, that is to say, to reduce the equations, if possible, until one term is left equal to one term; but I will show you later how, in the case also where two terms are left equal to one term, such a problem is solved."

So far as we know this promise was never fulfilled.

An equation of the first type is presented by the sixth problem of the sixth book, and this we reproduce from Heath: ²

"6. To find a right-angled triangle such that the area added to one of the perpendiculares makes a given number.

"Given number 7, triangle (3 x, 4 x, 5 x).

"Therefore 6 x² + 3 x = 7."

"In order that this might be solved it would be necessary that (half coefficient of x)² + product of coefficient of x² and absolute term should be a square: but (1 ½)² + 6 · 7 is not a square. Hence we must find, to replace (3, 4, 5), a right-angled triangle such that"

"(½ one perpendicular)² + 7 times area = a square;"

and the subsequent work leads to the equation, 84 x² + 7 x = 7, x = 4; and the solution (6, 7, 2 ½).

In the following problem (VI. 7) the value of x is given as ¼ for the equation

84 x² − 7 x = 7,

which equation is of the third type when the negative term is transposed after the manner of Diophantus. The equations

630 x² + 73 x = 6,
630 x² − 73 x = 6,
630 x² + 81 x = 4,
630 x² − 81 x = 4,

and

occur in the next four problems (VI. 8–11). Another problem of the third kind (IV. 39) is of especial interest because the rule is given for solving this type: ³

"To find three numbers such that the difference of the greatest and the middle has to the difference of the middle and the least a given ratio, and also the sum of any two is a square."

¹ Heath, Diophantus, p. 131.
² Heath, Diophantus, pp. 228–229.
³ Heath, Diophantus, pp. 197–198.
The discussion leads to the inequality, $2m^2 > 6m + 18$, which, since only integral solutions are desired, explains the use of 7 as the approximate square root of 45, in the following paragraph:

"When we solve such an equation, we multiply half the coefficient of $x$ into itself,—this gives 9,—then multiply the coefficient of $x^2$ into the units,—$2 \cdot 18 = 36$, —add this last number to the 9, making 45, and take the side [square root] of 45, which is not less than 7; add half the coefficient of $x$, —making a number not less than 10,—and divide the result by the coefficient of $x^2$; the result is not less than 5."

Of the second type, $ax^2 + e = bx$, Diophantus gives several illustrations, requiring frequently only the approximate value of the root. Problems of this kind are the following:

$$72m > 17m^2 + 17, \quad m \text{ not greater than } 4\frac{7}{4},$$  \hspace{1cm} (V. 10)
$$19m^2 + 19 > 72m, \quad m \text{ not less than } 6\frac{9}{10},$$  \hspace{1cm} (V. 10)
$$m^2 + 60 > 22m, \quad m \text{ not less than } 19,$$  \hspace{1cm} (V. 30)
$$m^2 + 60 < 24m, \quad m \text{ not greater than } 21,$$  \hspace{1cm} (V. 30)
and
$$172x = 336x^2 + 24,$$  \hspace{1cm} (VI. 22)

of which the statement is made that the root is not rational, and in the same problem

$$78848x^2 - 8432x + 225 = 0,$$

which has the rational root $4\frac{8}{14}$.

Commentaries on the Arithmetica began to appear very early. Probably the most interesting commentary from the modern point of view was the one written in the late fourth or early fifth century by Hypatia, the daughter of Theon of Alexandria. Unfortunately her writings are all lost, although there is ground for the belief\(^1\) that some remarks made by Michael Psellus (eleventh century) concerning Egyptian arithmetic and algebra were based on her commentary. She came naturally by her mathematical ability; her father Theon wrote a commentary on Ptolemy's Almagest and makes in this the earliest known reference to Diophantus.

Cossali,\(^2\) writing in 1797 on the history of algebra, conjectures that the step from the geometrical to the analytical form of solution took place in the period between Euclid and Diophantus.

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Now the Arabic Book of Chronicles\(^1\) (987 A.D.) states that the astronomer Hipparchus (second century B.C.) wrote a treatise on algebra, and Cantor\(^2\) inclines to the belief that there actually was such a work. No trace, however, has been found of it, and the probability is that Hipparchus did not write any systematic treatise on algebra or on quadratic equations. The word “algebra” indeed is Arabic in its origin and the use of it as a title goes back only to the time of our author, Mohammed Ibn Musa. Nevertheless it is possible that Greek mathematicians of the time of Hipparchus did occupy themselves with problems of the kind in question, because this was a natural development out of the consideration of rational right triangles, as given by Pythagoras and Plato, in connection with the geometrical treatment of quadratic equations as given by Euclid. Quadratic equations connect even more directly with the application of areas, of Pythagorean origin, which is extensively treated by Euclid.

Some two centuries after the period of Diophantus, Aryabhata, one of the earliest Hindu mathematicians of prominence, was born (476 A.D.). In the work of Aryabhata as presented by Rodet\(^3\) we find the solution of a quadratic equation assumed in the rule for finding the number of terms of an arithmetic series when the sum, difference, and first term are given. Nor does Aryabhata in India stand alone in the study of analysis, as Diophantus does in Greece. Brahmagupta of Ujjain, the centre of Indian learning, wrote on algebra in the early part of the seventh century and gave a rule\(^4\) for the solution of quadratic equations:

“To the absolute number multiplied by the [coefficient of the] square, add the square of half the [coefficient of the] unknown, the square root of the sum, less half the [coefficient of the] unknown, being divided by the [coefficient of the] square, is the unknown.”

In formula this corresponds to the solution

\[
x = \frac{\sqrt{(b/2)^2 + ac} - b/2}{a}
\]

\(^{1}\) Das Mathematiker-Verzeichnis im Führst des Ibn Abi Ja'far an-Nadim, translated by H. Suter, Abhandl. z. Geschichte der Mathematik, Vol. 6 (Leipzig, 1892), p. 22, and note, pp. 54-55. Suter holds that there is some error in the text, and this seems probable.


of the equation,

\[ ax^2 + bx = c. \]

Contemporary with Al-Khowarizmi is the Hindu writer Mahaviracarya, whose arithmetical and algebraical work has been translated into English by M. Raṅgacarya.\(^1\) Rules are given in the sections devoted to algebra for the three types of complete quadratic equations. A peculiarity of the treatment is that the unknown quantity and the square root of the unknown appear, rather than the unknown and its square. The significance of the work is that it shows a persistence of interest in algebra in India from the time of Aryabhata. Three centuries later Bhaskara (b. 1114 A.D.), another Hindu mathematician, made important contributions to the advance of the science.

The brief survey which we have given of the study of algebra before the time of Mohammed ibn Musa does not at all purpose to present the sources from which the great Arab drew his inspiration. Greece undoubtedly took mathematical ideas from Egypt, as Rodet\(^2\) some years ago pointed out with reference to algebra. Even more definite evidence is presented by the Greek use of unit fractions as well as by the references to Egyptian mathematics which were made by Plato and Herodotus, and much later by Michael Psellus. Babylon and Greece were constantly exchanging ideas;\(^3\) a striking proof of this is the Greek use of sexagesimal fractions. India, too, was not out of touch with these neighbors to her west. Especially in the fields of religion and the closely associated astrology we have abundant evidence not only of interchange of ideas between the East and the West but also of the recurrence in mediæval times of ideas advanced by more ancient civilizations. Yet we need to notice that we are dealing with the independent appearances of algebraic ideas, and that the mathematics of Egypt, Babylon, China, Greece, and India was developing from within. Algebra is not, as often assumed, an

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3 F. Cunet, *Babylon und die griechische Astronomie. Neues Jahrbücher für das klassische Altertum* ... Vol. 27 (1911), pp. 1–10; *The Oriental Religions in Roman Paganism* (Chicago, 1911); and *Astrology and Religion among the Greeks and Romans* (New York, 1912).
artificial effort of human ingenuity, but rather the natural expression of man’s interest in the numerical side of the universe of thought. Tables of square and cubic numbers in Babylon; geometric progressions, involving the idea of powers, together with linear and quadratic equations in Egypt; the so-called Pythagorean theorem in India, and possibly in China, before the time of Pythagoras; and the geometrical solution of quadratic equations even before Euclid in Greece, are not isolated facts of the history of mathematics. While they do indeed mark stages in the development of pure mathematics, this is only a small part of their significance. More vital is the implication that the algebraical side of mathematics has an intrinsic interest for the human mind not conditioned upon time or place, but dependent simply upon the development of the reasoning faculty. We may say that the study of powers of numbers, and the related study of quadratic equations, were an evolution out of a natural interest in numbers; the facts which we have presented are traces of the process of this evolution.